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On zero-sum 6-flows of graphs

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ABSTRACT

For a graph G, a *zero-sum flow* is an assignment of non-zero real numbers on the edges of G such that the total sum of all edges incident with any vertex of G is zero. A *zero-sum k-flow* for a graph G is a zero-sum flow with labels from the set $\{\pm 1, \ldots, \pm(k-1)\}$. In this paper for a graph G, a necessary and sufficient condition for the existence of zero-sum flow is given. We conjecture that if G is a graph with a zero-sum flow, then G has a zero-sum 6-flow. It is shown that the conjecture is true for 2-edge connected bipartite graphs, and every r-regular graph with r even, $r > 2$, or $r = 3$.

1. Introduction

Let G be a directed graph. A *k-flow* on G is an assignment of integers with maximum absolute value $k-1$ to each edge of G such that for any vertex of G, the sum of the labels on incoming edges is equal to that of the labels on outgoing edges. A *nowhere-zero k-flow* is a k-flow with no zeros.

A celebrated conjecture of Tutte says that,

Conjecture (TC) (*Tutte's 5-flow Conjecture* [8]). Every bridgeless graph has a nowhere-zero 5-flow.

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Two well-known results on this conjecture are the following:

Theorem A [4]. *Every bridgeless graph has a nowhere-zero* 8-*flow*.

Theorem B [7]. *Every bridgeless graph has a nowhere-zero* 6-*flow*.

For a thorough account on the above conjecture and subsequent results, see [5,11]. In this paper we have chosen a linear algebraic approach to look at Tutte's Conjecture which provides a motivation to adopt a similar conjecture for undirected graphs.

For a directed graph G of order n and size m, the incidence matrix $W(G) = (w_{ij})$ is an $n \times m$ matrix whose rows are indexed by the set of vertices of G, $\{v_1, \ldots, v_n\}$, and columns are indexed by the set of edges of G, $\{e_1, \ldots, e_m\}$, defined by

$$w_{ij} = \begin{cases} +1 & \text{if } e_j \text{ is an incoming edge to } v_i, \\ -1 & \text{if } e_j \text{ is an outgoing edge from } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for an undirected graph G, the incident matrix of G, W, is defined as follows:

$$w_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

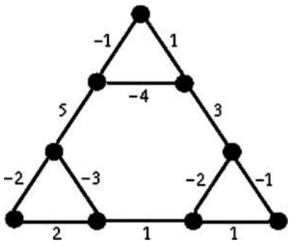
For a graph G, a *zero-sum flow* is an assignment of non-zero real numbers on the edges of G such that the total sum of the assignments of all edges incident with any vertex of G is zero. A *zero-sum k-flow* for a graph G is a zero-sum flow with labels from the set $\{\pm 1, \ldots, \pm(k-1)\}$.

Clearly, there is a one to one correspondence between a flow of a graph G and an element of the null space of $W(G)$. Indeed, if $[c_1, \ldots, c_m]^T$ is an element of the null space of G, then we assign the value c_i to e_i and consequently we obtain a flow. This is certainly a linear algebraic approach to the "flow problem". By this token a nowhere-zero flow is an element of null space. Therefore, in the language of linear algebra, Tutte's Conjecture says that if G is a graph with no cut edge, then there is an integral nowhere-zero element in the null space of the incidence matrix of every orientation of G for which the absolute value of each entry does not exceed 4. This linear algebraic approach to TC allows us to formulate the following conjecture:

Let G be a simple graph with incidence matrix W. If there exists a nowhere-zero real element of the null space of W, then there is an integral nowhere-zero element in the null space of W for which the absolute value of each entry does not exceed 5, or equivalently,

Zero-Sum Conjecture (ZSC). If G is a graph with a zero-sum flow, then G has a zero-sum 6-flow.

Remark 1. There are some graphs which have zero-sum 6-flows but have no zero-sum 5-flows. The following example with 9 vertices was discovered through an exhaustive search [6]. Interestingly enough among the graphs with at most nine vertices, this is the only graph with this property.



Let G be a graph and $v \in V(G)$. We say that *zero-sum rule* holds on v, when the sum of assignments of all edges incident with v is zero. In this paper we show that ZSC is true for 2-edge connected bipartite graphs and k-regular graphs, k even or $k = 3$.

A *bidirected* graph G is a directed graph with vertex set $V(G)$ and edge set $E(G)$ such that each edge-vertex incidence is assigned an orientation, either out of or into the vertex. Thus every edge is oriented with one of the four possible orientations. An integer-valued function f on $E(G)$ is a nowhere-zero bidirected k-flow if for every $e \in E(G)$, $0 < |f(e)| < k$, and at every vertex v, the value of f-flow in equals the amount of f-flow out. Bouchet conjectured that every bidirected graph which admits a nowhere-zero bidirected flow admits a nowhere-zero bidirected 6-flow, see [2]. This conjecture has been verified by Bouchet if 6 is replaced by 216, and by Zyka if 6 is replaced by 30, see [12]. Indeed a zero-sum k-flow of G is exactly a nowhere-zero k-flow in the bidirected graph G, where each edge is oriented as:

Therefore ZSC is a special case of Bouchet's Conjecture.

2. More on zero-sum conjecture

We begin this section with the following lemma.

Lemma 1. *Let G be a 2-edge connected bipartite graph. Then G has a zero-sum 6-flow.*

Proof. Let G be 2-edge connected bipartite graph. By Theorem B, every 2-edge connected bipartite graph has a nowhere-zero 6-flow. Without loss of generality one may assume that the direction of all edges is from one part of G to the other part. Now, by removing the direction of all edges we conclude that in any vertex the zero-sum rule holds and hence the proof is complete. □

The next theorem presents a geometric interpretation for graphs having a zero-sum flow.

Theorem 1. *Suppose G is not a bipartite graph. Then G has a zero-sum flow if and only if for any edge e of G, $G \setminus \{e\}$ has no bipartite component.*

Proof. With no loss of generality, assume that G is a connected graph. First suppose that G has a zero-sum flow. We consider two cases:

Case 1. Assume that $G\setminus\{e\}$ has two components H_1 and H_2. Let H_i be of order n_i and size m_i for $i = 1, 2$. Suppose that H_1 is bipartite. Then $\mathrm{nul}(H_1) = m_1 - n_1 + 1$ (Exercise 7, p. 37, [1]), where by $\mathrm{nul}(H_1)$, we mean the dimension of the null space of the incidence matrix of H_1. Also we have $\mathrm{nul}(H_2) \geqslant m_2 - n_2$. Thus

$$\mathrm{nul}(G \setminus \{e\}) = \mathrm{nul}(H_1) + \mathrm{nul}(H_2) \geqslant m_1 + m_2 - (n_1 + n_2) + 1 = m - n,$$

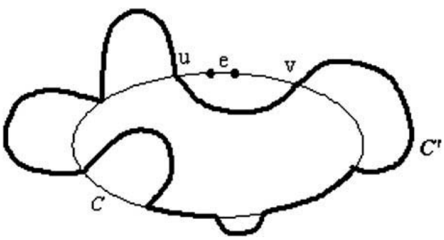
where n and m are the order and the size of G, respectively. On the other hand $m - n = \mathrm{nul}(G) \geqslant \mathrm{nul}(G\setminus\{e\})$. Therefore $\mathrm{nul}(G\setminus\{e\}) = m - n$. Since $\mathrm{nul}(G) = m - n$, we conclude that each element of the null space of G is zero on e, which is a contradiction.

Case 2. Let $G\setminus\{e\}$ be a connected graph. Suppose that $G\setminus\{e\}$ is a bipartite graph. Then $\mathrm{nul}(G\setminus\{e\}) = (m - 1) - n + 1 = m - n$. Now, since $\mathrm{nul}(G) = m - n$, as in Case 1, this is a contradiction.

To prove the sufficiency, first we show that for any edge e, there exists an element in the null space of G whose corresponding entry with e is non-zero.

If e lies on an even cycle, then the assertion is immediate. If e is a cut edge of G, then since none of the components of $G\setminus\{e\}$ is not bipartite, hence G contains two odd cycles joining by a path P containing e. We assign 2 and -2 to the edges of the path, alternatively. If the label of an ending edge of the path is 2 (-2), then we assign -1 (1) to the two edges of the cycle adjacent to this edge of the path, and the remaining edges of the cycle are labeled by 1 (-1) and -1 (1), alternatively. Thus there exists an element in the null space of the incidence matrix of G whose entry corresponding to e is non-zero. Next, assume that e does not lie on any even cycle but on some odd cycle, say, $\mathscr{C}$. Now, we claim that there is an odd cycle $\mathscr{C}_1$ in G containing e such that $|V(\mathscr{C}) \cap V(\mathscr{C}_1)| \leqslant 1$. If the claim is proved, then we obtain two odd cycles joining a path P which one of them contains e. Note that in the case $|V(\mathscr{C}) \cap V(\mathscr{C}_1)| = 1$, the length of P is zero and the previous method works for showing that the existence of an element in the null space of the incidence matrix of G whose entry corresponding to e is non-zero.

Since $G \setminus \{e\}$ is not bipartite, there exists an odd cycle $\mathscr{C}'$ not containing e. Suppose that $|V(\mathscr{C}') \cap V(\mathscr{C})| \geqslant 2$. Let $u, v \in V(\mathscr{C}) \cap V(\mathscr{C}')$. Assume that the arc uv is a path on $\mathscr{C}$ which contains e and $V(uv) \cap V(\mathscr{C}) \cap V(\mathscr{C}') = \{u, v\}$. Since $\mathscr{C}'$ is an odd cycle, then the two paths on $\mathscr{C}'$, determined by u and v, have different parities. Therefore there exists an even cycle containing e which is a contradiction. Thus $|V(\mathscr{C}) \cap V(\mathscr{C}_1)| \leqslant 1$ and the claim is proved. Let $e \in E(G)$ and W_e, be the set of all vectors in the null space of the incidence matrix of G whose corresponding entries of e are zero. Clearly, W_e is a vector space. Let W be the null space of the incidence matrix of G. If $W \subseteq \bigcup_{e \in E(G)} W_e$, then since W is a vector space over the infinite field $\mathbb{R}$, it is well-known that there exists some edge e such that $W \subseteq W_e$, which is a contradiction. Thus there exists an element $\alpha \in W \setminus \bigcup_{e \in E(G)} W_e$ and the proof is complete. $\square$



Before proving the next lemma we need the following result.

Let G be a 3-regular graph with at most two cut edges, then G has a 1-factor, [3].

Lemma 2. *Let G be a graph with no connected component, K_2, such that for every $v \in V(G)$, $d(v) \in \{1, 3\}$. Suppose that the induced subgraph on all vertices of degree 3 has no cut edge. If h is a pendant edge of G with a value from $\{-2, 4\}$, then there exists a function f on $E(G)$ which agrees on h and has the following properties*:

(i) *For all $e \in E(G)$, $f(e) \in \{1, -2, 4\}$.*
(ii) *For every $v \in V(G)$ of degree 3, the zero-sum rule holds.*
(iii) *If $e \in E(G)$ is a pendant edge, then $f(e) \in \{-2, 4\}$.*

Proof. Consider two copies of G, say G_1 and G_2. Assume that u_iv_i, $1 \leqslant i \leqslant k$ are all the edges of G_1, such that $u_i, v_i \in V(G_1)$, $d(v_i) = 1$. Also, suppose that u'_i and v'_i are vertices corresponding to u_i and v_i $(i = 1, \ldots, k)$ in G_2, respectively. Let G^* be a graph obtained by removing the vertices $v_1, \ldots, v_k$ and $v'_1, \ldots, v'_k$ and joining u_i and u'_i in $G_1 \cup G_2$, for $i = 1, \ldots, k$. Since none of the connected components of G is K_2, G^* is a 3-regular graph.

First we assume that G^* has exactly one cut edge h with value -2. By [3], G^* has a 1-factor M.

Since $G^* \setminus E(M)$ is 2-regular, M contains h. For every $e \in E(M)$, let $f(e) = -2$ and for every $e \in E(G^*) \setminus E(M)$, let $f(e) = 1$. Now, assume that the value of h is 4. In this case first we apply the same procedure and then we multiply each value by -2.

Next, suppose that G^* has no cut edge. Let t be an edge incident with h. Then by a result in [9], G^* has a 1-factor M containing t. If the value of h is -2, then we assign $f(e) = 4$ to any edge of M, and for any edge $e \in E(G^*) \setminus E(M)$, we assign $f(e) = -2$. If the value of h is 4, then G^* has a 1-factor M containing h. If $e \in M$, let $f(e) = 4$ and if $e \in E(G^*) \setminus E(M)$, let $f(e) = -2$. Now, by restricting to the edges of G, we obtain the result.

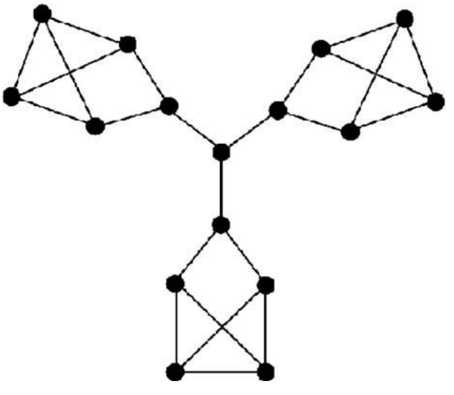
Finally, we note that if G has at least two cut edges, then G^* has no cut edge and this completes the proof. $\square$

Theorem 2. *Every 3-regular graph has a zero-sum 5-flow.*

Proof. Indeed we obtain a stronger result, namely every 3-regular graph G has a zero-sum flow with values from $\{1, -2, 4\}$ and the value of every cut edge is -2 or 4. To show this we construct a rooted tree T from G where every maximal 2-edge connected subgraph of G is considered as a vertex of T and $E(T)$ consists of all the cut edges of G. Now, by traversing T, level by level, we find a zero-sum 5-flow for G. We start from the root of T, H. By Lemma 2, we assign an element of $\{1, -2, 4\}$ to each edge of H and -2 or 4 to every cut edge of G incident with H such that for every $v \in V(H)$ the zero-sum rule holds. Now, we move to the next vertex level of T. Let H' be a vertex adjacent to H in T. At this stage of the process there exists just one cut edge of G incident with H' which is labeled by -2 or 4.

Now, by employing Lemma 2, we label the edges of H' and apply the same procedure as in H. By continuing this procedure we obtain a zero-sum 5-flow for G, as desired. □

Remark 2. The following graph shows that in the above theorem, zero-sum 5-flow can not be replaced with a zero-sum 4-flow. To see this we assume to the contrary and let G have a zero-sum 4-flow. Since the sum of three odd numbers is odd, therefore in any vertex an even value should appear. On the other hand two even values can not be assigned to the edges incident with each vertex. This implies that the edges with even values in G form a 1-factor and this is a contradiction, because it is a routine work to check that G does not contain a 1-factor.



Theorem 3. *Let r be even, and $r \geqslant 4$. Then every r-regular graph, has a zero-sum 3-flow.*

Proof. Let G be an r-regular graph with r even. In this case by Theorem 3.3.9 of [10], G is 2-factorable. If the number of 2-factors are even, then we assign value 1 to the edges of one half of 2-factors and -1 to the edges of the other half. If the number of 2-factors is odd, then we choose three 2-factors, F_1, F_2 and F_3 and assign 2 to the edges of F_1 and assign -1 to the edges of F_2 and F_3. Now, the number of the remaining 2-factors is even and we return to the beginning. □

3. Nowhere-zero k-flows and zero-sum k-flows

In this section we establish a relation between a zero-sum k-flow and a nowhere-zero k-flow for a graph. More precisely, to every graph G we associate a new graph such that the existence of a nowhere-zero k-flow for G is equivalent to the existence of a zero-sum k-flow for the new graph. To do this we need some definitions.

Let G be a graph, then $\mathbf{S(G)}$ is a graph obtained from G by augmenting exactly one new vertex on each edge of G. The following lemma has a simple proof and so the proof is omitted.

Lemma 3. *A graph G has a nowhere-zero k-flow if and only if $S(G)$ has a zero-sum k-flow.*

We call a graph a $(2, 3)$-*graph* if the degree of each vertex is 2 or 3.

Theorem 4. *If ZSC is true for any* $(2, 3)$-*graph*, *then it is true for any graph*.

Proof. Suppose that G is equipped with a zero-sum flow. Let v be a vertex of G with degree at least 4. Hence there are at least two edges, say vu_1 and vu_2, with values of the same sign. We replace v with two new vertices v_1 and v_2 and join v_1 to u_1 and u_2 and join v_2 to the rest of the vertices adjacent to v. Now, we add another vertex w and join w to v_1 and v_2. By pursuing this process we get a $(2, 3)$-bi-regular graph K. By assumption K has a zero-sum 6-flow which implies that G has a zero-sum 6-flow. $\square$

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